Static dynamic characterstic

modeled by a constant-coefficient linear differential equation

$$a_k \frac{d^k y(t)}{dt^k} + \cdots a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = x(t)$$

Think of the Laplace domain as an extension of the Fourier transform

- Fourier analysis is restricted to sinusoidal signals
  - x(t) = sin(ωt) = e−jωt
- Laplace analysis can also handle exponential behavior
  - x(t) = e−σtSin(ωt) = e−(σ +jω)

Laplace transform of a time signal y(t) is denoted by

L[y(t)] = Y(s)

The fundamental relationship is the one that concerns the transformation of differentiation

$$L\left[\frac{d}{dt} y(t)\right] = sY(s) - f(0)$$

Other useful relationships are

$$\text{Impulse}: \; L[\delta(t)] = 1 \qquad \text{Decay}: \; L[\exp(at)] = (s-a)^{-1}$$

$$\text{Step}: \; L[u(t)] = \frac{1}{s} \qquad \text{Sine}: \; L[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\text{Ramp}: \; L[r(t)] = \frac{1}{s^2} \qquad \text{Cosine}: \; L[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$$

Applying the Laplace transform to the sensor model yields

$$L\left[a_k \frac{d^k y}{dt^k} + \cdots a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y(t) = x(t)\right]$$

$$\Downarrow$$

$$(a_k s^k + \cdots a_2 s^2 + a_1 s + a_0) Y(s) = X(s)$$

$$\Downarrow$$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{a_k s^k + \cdots a_2 s^2 + a_1 s + a_0}$$

Zero-order is the desirable response of a sensor

$$y(t) = k \cdot x(t) \Rightarrow \frac{Y(s)}{X(s)} = k$$

First-order sensors

$$a_1 \frac{dy}{dt} + a_0 y(t) = x(t) \Rightarrow \frac{Y(s)}{X(s)} = \frac{1}{a_1 s + a_0} = \frac{k}{\tau s + 1}$$

Step response

- y(t) = Ak(1−exp(−t/τ))
  - A is the amplitude of the step
  - k (=1/a0) is the static gain, which determines the static response
  - τ (=a1/a0) is the time constant, which determines the dynamic response

Ramp response

- y(t) = Akt − Akτu(t) + Akτ exp(−t/τ)

Frequency response

- Better described by the amplitude and phase shift plots

$$M(\ddot{x}_i - \ddot{x}_0) = Kx_0 + B\dot{x}_0$$

$$\Downarrow$$

$$Ms^2 X_i(s) = X_0(s)\left[K + Bs + Ms^2\right]$$

$$\Downarrow$$

$$\frac{X_0(s)}{s^2 X_i(s)} = \frac{M}{K} \frac{K/M}{s^2 + s(B/M) + K/M}$$

First Order Systems: frequency response

The complete solution: $y(t) = Ce^{-t/\tau} + \frac{KA}{\sqrt{1+(\omega\tau)^2}} \sin(\omega t - \tan^{-1}\omega\tau)$

Transient response — Steady state response = Frequency response

If we do interest in only steady state response of the system, we can write the equation in general form

$$y(t) = Ce^{-t/\tau} + B(\omega)\sin[\omega t + \phi(\omega)]$$

$$B(\omega) = \frac{KA}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

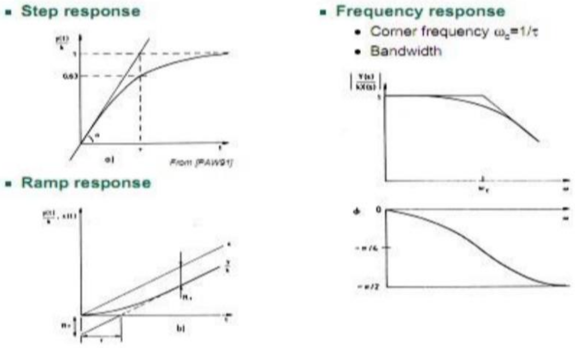
$$\phi(\omega) = -\tan^{-1}\omega\tau$$

Where $B(\omega)$ = amplitude of the steady state response and $\phi(\omega)$ = phase shift

$$M(\omega) = \frac{B}{KA} = \frac{1}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

The amplitude ratio $M(\omega) = \frac{1}{\sqrt{(\omega\tau)^2+1}}$ The phase angle is $\phi(\omega) = -\tan^{-1}(\omega\tau)$

Dynamic error, $\delta(\omega) = M(\omega) - 1$: a measure of an inability of a system to adequately reconstruct the amplitude of the input for a particular frequency



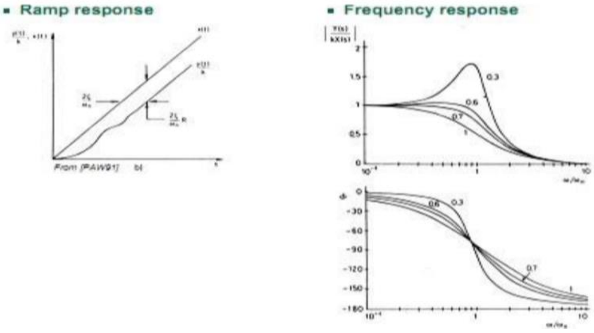
Second-order sensors

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y(t) = x(t) \Rightarrow \frac{Y(s)}{X(s)} = \frac{1}{a_2 s^2 + a_1 s + a_0}$$

Second-order transfer function

$$\frac{Y(s)}{X(s)} = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\text{with} \quad k = \frac{1}{a_0}, \quad \zeta = \frac{a_1}{2\sqrt{a_0 a_1}}, \quad \omega_n = \sqrt{\frac{a_0}{a_2}}$$



The equilibrium equation is:

n^{th} Order ordinary linear differential equation with constant coefficient

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m x(t)}{dt^m} + b_{m-1}\frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$F(t)$ = forcing function

The solution $y(t) = y_{ocf} + y_{opi}$

The solution y_{ocf} is obtained by calculating the n roots of the algebraic *characteristic equation*

Characteristic equation $a_n D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0 = 0$

Roots of the characteristic equation: $D = s_1, s_2, \ldots, s_n$

Complementary-function solution:

1. Real roots, unrepeated: Ce^{st}
2. Real roots, repeated: each root s which appear p times $(C_0 + C_1 t + C_2 t^2 + \ldots + C_{p-1}t^{p-1})e^{st}$
3. Complex roots, unrepeated: the complex form: $a \pm ib$ $Ce^{at}\sin(bt+\phi)$
4. Complex roots, repeated: each pair of complex root which appear p times $[C_0\sin(bt+\phi_0) + C_1 t\sin(bt+\phi_1) + C_2 t^2 \sin(bt+\phi_2) + \ldots + C_{p-1}t^{p-1}\sin(bt+\phi_{p-1})]e^{at}$

Method of undetermined coefficients:

$$y_{opi} = Af(t) + Bf'(t) + Cf''(t) + \ldots$$

Zero order system

All the a's and b's other than a_0 and b_0 are zero.

$$a_0 y(t) = b_0 x(t) \longrightarrow y(t) = Kx(t)$$

where K = static sensitivity = b_0/a_0

$$V = V_r \cdot \frac{x}{x_m} \quad \text{here, } K = V_r / x_m$$

Where $0 \le x \le x_m$ and V_r is a reference voltage

First order system

All the a's and b's other than a_1, a_0 and b_0 are zero.

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t)$$

$$\tau \frac{dy(t)}{dt} + y(t) = Kx(t) \Longleftrightarrow \frac{y}{x}(D) = \frac{K}{\tau D + 1}$$

Where $K = b_0/a_0$ is the static sensitivity

$\tau = a_1/a_0$ is the system's time constant (dimension of time)

First order system: step response

Assume for $t < 0$, $y = y_0$, at time = 0 the input quantity, x increases instantly by an amount A. Therefore $t > 0$

$$x(t) = AU(t) = \begin{cases} 0 & t \le 0 \\ A & t > 0 \end{cases}$$

$$\tau \frac{dy(t)}{dt} + y(t) = KAU(t)$$

complete solution:

$$y(t) = \underbrace{Ce^{-t/\tau}}_{y_{ocf}} + \underbrace{KA}_{y_{opi}}$$

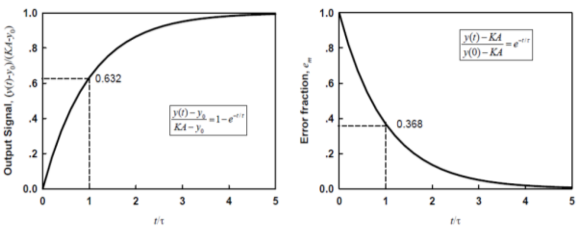
y_{ocf} Transient response; y_{opi} Steady state response

Applying the initial condition, we get $C = y_0 - KA$, thus gives

$$y(t) = KA + (y_0 - KA)e^{-t/\tau}$$

Here, we define the term error fraction as

$$e_m(t) = \frac{y(t) - KA}{y_0 - KA} = \frac{y(t) - y(\infty)}{y(0) - y(\infty)} = e^{-t/\tau}$$



Non-dimensional step response of first-order instrument

Determination of time constant

$$e_m = \frac{y(t) - KA}{y(0) - KA} = e^{-t/\tau} \qquad \ln e_m = 2.3 \log e_m = -\frac{t}{\tau}$$

Slope = $-1/\tau$

First order systems: Ramp Response

Assume that at initial condition, both y and x = 0, at time = 0, the input quantity start to change at a constant rate $\dot{q}_u$ Thus, we have

$$x(t) = \begin{cases} 0 & t \le 0 \\ \dot{q}_u t & t > 0 \end{cases}$$

Therefore

$$\tau \frac{dy(t)}{dt} + y(t) = K\dot{q}_u t U(t)$$

The complete solution:

$$y(t) = \underbrace{Ce^{-t/\tau}}_{\text{Transient response}} + \underbrace{K\dot{q}_u(t - \tau)}_{\text{Steady state response}}$$

Applying the initial condition, gives

$$y(t) = K\dot{q}_u(\tau e^{-t/\tau} + t - \tau)$$

Measurement error

$$e_m = x(t) - \frac{y(t)}{K} = \underbrace{-\dot{q}_u \tau e^{-t/\tau}}_{\text{Transient error}} + \underbrace{\dot{q}_u \tau}_{\text{Steady state error}}$$

First order systems: frequency response

From the response of first-order system to sinusoidal inputs, $x(t) = A\sin\omega t$ we have

$$\tau \frac{dy}{dt} + y = KA\sin\omega t \Longleftrightarrow (\tau D + 1)y(t) = KA\sin\omega t$$

The complete solution:

$$y(t) = \underbrace{Ce^{-t/\tau}}_{\text{Transient response}} + \underbrace{\frac{KA}{\sqrt{1+(\omega\tau)^2}}\sin(\omega t - \tan^{-1}\omega\tau)}_{\text{Steady state response} \;=\; \text{Frequency response}}$$

If we do interest in only steady state response of the system, we can write the equation in general form

$$y(t) = Ce^{-t/\tau} + B(\omega)\sin[\omega t + \phi(\omega)]$$

$$B(\omega) = \frac{KA}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

$$\phi(\omega) = -\tan^{-1}\omega\tau$$

Where $B(\omega)$ = amplitude of the steady state response and $\phi(\omega)$ = phase shift

$$M(\omega) = \frac{B}{KA} = \frac{1}{\left[1+(\omega\tau)^2\right]^{1/2}}$$

The amplitude ratio $M(\omega) = \frac{1}{\sqrt{(\omega\tau)^2 + 1}}$ The phase angle is $\phi(\omega) = -\tan^{-1}(\omega\tau)$

Dynamic error, $\delta(\omega) = M(\omega) - 1$: a measure of an inability of a system to adequately reconstruct the amplitude of the input for a particular frequency

Second Order Systems

In general, a second-order measurement system subjected to arbitrary input, $x(t)$

$$a_2 \frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t) \Rightarrow \left(\frac{D^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}D + 1\right)y(t) = Kx(t)$$

$$\frac{1}{\omega_n^2}\frac{d^2y(t)}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy(t)}{dt} + y(t) = Kx(t)$$

The essential parameters

$K = \frac{b_0}{a_0}$ = the static sensitivity

$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}}$ = the damping ratio, dimensionless

$\omega_n = \sqrt{\frac{a_0}{a_2}}$ = the natural angular frequency

Consider the characteristic equation

$$\frac{1}{\omega_n^2}D^2 + \frac{2\zeta}{\omega_n}D + 1 = 0$$

This quadratic equation has two roots:

$$S_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Depending on the value of ζ, three forms of complementary solutions are possible

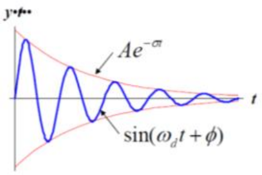
Overdamped ($\zeta > 1$): $y_{oc}(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n t}$

Critically damped ($\zeta = 1$): $y_{oc}(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$

Underdamped ($\zeta < 1$): : $y_{oc}(t) = Ce^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \Phi\right)$

Case 1 Underdamped ($\zeta < 1$):

$$S_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = \sigma \pm j\omega_d$$

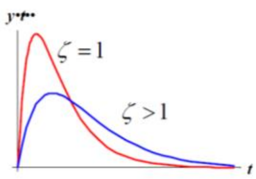


Case 2 Overdamped ($\zeta > 1$):

$$S_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n$$

Case 3 Critically damped ($\zeta = 1$):

$$S_{1,2} = -\omega_n$$



Second Order Systems: Step Response

For a step input $x(t)$ $\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=KAU(t) \Leftrightarrow \left(\frac{D^2}{\omega_n^2}+\frac{2\zeta}{\omega_n}D+1\right)y(t)=KAU(t)$

With the initial conditions: $y = 0$ at $t = 0+$, $dy/dt = 0$ at $t = 0+$

The complete solution:

Overdamped ($\zeta > 1$): $\frac{y(t)}{KA}=-\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+\frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}+1$

Critically damped ($\zeta = 1$): $\frac{y(t)}{KA}=-(1+\omega_n t)e^{-\omega_n t}+1$

Underdamped ($\zeta < 1$): $\frac{y(t)}{KA}=-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\phi\right)+1 \qquad \phi=\sin^{-1}\left(\sqrt{1-\zeta^2}\right)$

Ringing period $T_d = \frac{2\pi}{\omega_d}$

Ringing frequency: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Second Order Systems: Ramp Response

For a ramp input $x(t)=\dot{q}_{is}tU(t)$ $\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=K\dot{q}_{is}tU(t)$

With the initial conditions: $y = dy/dt = 0$ at $t = 0+$

The possible solutions:

Overdamped: $\frac{y(t)}{K}=\dot{q}_{is}t-\frac{2\zeta\dot{q}_{is}}{\omega_n}\left(1+\frac{2\zeta^2-1-2\zeta\sqrt{\zeta^2-1}}{4\zeta\sqrt{\zeta^2-1}}e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+\frac{-2\zeta^2+1-2\zeta\sqrt{\zeta^2-1}}{4\zeta\sqrt{\zeta^2-1}}e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}\right)$

Critically damped: $\frac{y(t)}{K}=\dot{q}_{is}t-\frac{2\dot{q}_{is}}{\omega_n}\left[1-\left(1+\frac{\omega_n t}{1}\right)e^{-\omega_n t}\right]$

Underdamped: $\frac{y(t)}{K}=\dot{q}_{is}t-\frac{2\zeta\dot{q}_{is}}{\omega_n}\left[1-\frac{e^{-\zeta\omega_n t}}{2\zeta\sqrt{1-\zeta^2}}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t+\phi\right)\right] \qquad \phi=\tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}$

Steady state error = $\frac{2\dot{q}_{is}\zeta}{\omega_n}$ Steady state time lag = $\frac{2\zeta}{\omega_n}$

Second Order Systems: Frequency Response

The response of a second-order to a sinusoidal input of the form $x(t) = A\sin\omega t$

$$y(t)=y_{ocf}(t)+\frac{KA}{\left\{\left[1-(\omega/\omega_n)^2\right]^2+(2\zeta\omega/\omega_n)^2\right\}^{1/2}}\sin[\omega t+\phi(\omega)]$$

where $\phi(\omega)=-\tan^{-1}\frac{2\zeta}{\omega/\omega_n-\omega_n/\omega}$

The steady state response of a second-order to a sinusoidal input

$$y_{steady}(t)=B(\omega)\sin[\omega t+\phi(\omega)]$$

$$B(\omega)=\frac{KA}{\left\{\left[1-(\omega/\omega_n)^2\right]^2+(2\zeta\omega/\omega_n)^2\right\}^{1/2}} \qquad \phi(\omega)=-\tan^{-1}\frac{2\zeta}{\omega/\omega_n-\omega_n/\omega}$$

Where $B(\omega)$ = amplitude of the steady state response and $\phi(\omega)$ = phase shift

$$M(\omega)=\frac{B}{KA}=\frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2+(2\zeta\omega/\omega_n)^2\right\}^{1/2}}$$

The amplitude ratio

$$M(\omega)=\frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2+(2\zeta\omega/\omega_n)^2\right\}^{1/2}}$$

The phase angle

$$\phi(\omega)=-\tan^{-1}\frac{2\zeta}{\omega/\omega_n-\omega_n/\omega}$$

Second Order System

Rise time: $t_r=\frac{\arctan(-\omega_d/\delta)}{\omega_d}$

Maximum overshoot: $M_p=\exp\left(-\pi\zeta/\sqrt{1-\zeta^2}\right)$

Peak time: $t_p=\frac{\pi}{\omega_d}$

Resonance frequency: $\omega_r=\omega_n\sqrt{1-2\zeta^2}$

Resonance amplitude: $M_r=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$

where $\delta=\zeta\omega_n, \omega_d=\omega_n\sqrt{1-\zeta^2}$, and $\phi=\arcsin(\sqrt{1-\zeta^2})$

Dynamic Characteristics

H(s) = Y(s)/X(s)

Dynamic error, $\delta(\omega) = M(\omega) - 1$

$t_r \approx \frac{0.35}{f_c}$

$M(\omega)=\frac{1}{\sqrt{\omega^2\tau^2+1}}$

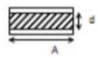
$\text{Dynamic error}=(M(\omega)-1)\times100\%=\left(\frac{1}{\sqrt{\omega^2\tau^2+1}}-1\right)\times100\%$

Phase shift $\phi=-\arctan\omega\tau$

Sensing Principles

- The capacitance of a parallel plate capacitor is

$C=\frac{\varepsilon_0\varepsilon_r A}{d}$



Strain gauges

- Strain is a fractional change ($\Delta L/L$) in the dimensions of an object as a result of mechanical stress (force/area)
- The resistance R of a strip of material of length L, cross-section A and resistivity ρ is $R=\rho L/A$
- Differentiating, the gauge factor G becomes

$$\frac{\Delta R}{R}=\frac{\Delta L}{L}-\frac{\Delta A}{A}+\frac{\Delta\rho}{\rho}\cong(1+2\nu)\frac{\Delta L}{L}+\frac{\Delta\rho}{\rho}\Rightarrow G=\frac{\Delta R/R}{\Delta L/L}=\underbrace{(1+2\nu)}_{\text{GEOMETRIC EFFECT}}+\underbrace{\frac{\Delta\rho}{\rho\Delta L}}_{\text{PIEZO-RESISTIVE EFFECT}}$$

relationship between temperature difference & output voltage of a thermocouple is nonlinear and is approximated by polynomial:

$$\Delta T=\sum_{n=0}^{N}a_n v^n.$$

NTC thermistor

$$R_T=R_0\exp\left[B\left(\frac{1}{T}-\frac{1}{T_0}\right)\right]$$

first-order, relationship between resistance and temperature is linear:

$$\Delta R = k\Delta T$$

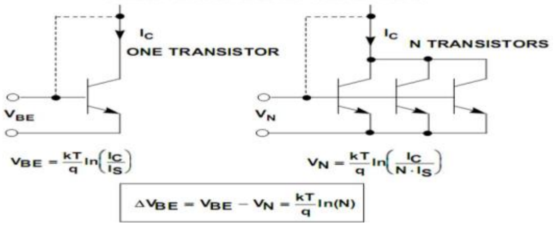
$$R_T=R_0\left[1+\alpha_1T+\alpha_2T^2+\cdots\alpha_nT^n+\right]\cong R_0\left[1+\alpha_1T\right]$$

All semiconductor temperature sensors make use of the relationship between a bipolar junction transistor's (BJT) base-emitter voltage to its collector current:

$$V_{BE}=\frac{kT}{q}\ln\left(\frac{I_C}{I_S}\right)$$

If we take N transistors identical to the first and allow the total current Ic to be shared equally among them, we find that the new base-emitter voltage is given by the equation

$$V_N=\frac{kT}{q}\ln\left(\frac{I_C}{N\cdot I_S}\right)$$



$$V_{PTAT} = \frac{2R1(V_{BE} - V_N)}{R2} = 2\frac{R1}{R2}\frac{kT}{q}\ln(N).$$

Capacitive-sensors

Capacitance (in farad, F) → $C = \frac{Q}{V}$ ← Charge (in coulomb, C); ← voltage difference between two plates (in volts, V).

force (in newton, N) → $V = \frac{Fd}{q} = Ed$ ← the distance between two plates

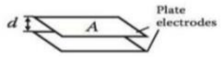
↑ electric field (in volt per meter, V · m⁻¹ or newton per coulomb, N · C⁻¹) between two parallel plates,

- The voltage–current relationship of a capacitor is expressed by

$$V(t) = \frac{1}{C}\int I(t)dt$$

- Capacitors in series: $\frac{1}{C_{eq}} = \sum \frac{1}{C_i}$
- Capacitors in parallel: $C_{eq} = \sum C_i$

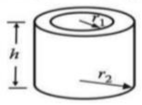
Parallel-(flat) plate capacitor



$$C = \frac{\varepsilon_0 \varepsilon_r A}{d}$$

Ratio A/d is called the *geometry factor* for a parallel-plate capacitor.

Cylindrical (coaxial) capacitor



$$C = \frac{2\pi\varepsilon_0\varepsilon_r h}{\ln(r_2/r_1)} \quad (h \gg r_2)$$

Ratio $2\pi h/\ln(r_2/r_1)$ is the *geometry factor* for a cylindrical capacitor.

Accelerometer

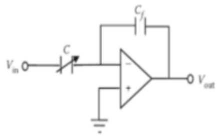
$$\Delta C \approx \frac{n\varepsilon_0\varepsilon_r l_f}{d_f}(2l_m + l_f)\theta$$

The level h of the dielectric material can be found by

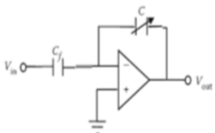
$$h = \frac{(C - C_0)\ln(r_2/r_1)}{2\pi\varepsilon_0(\varepsilon_r - 1)}$$

Amplifier

V_{out} proportional to A $\quad V_{out} = -\frac{C}{C_f}V_{in} = -\frac{\varepsilon_0\varepsilon_r A}{C_f d}V_{in}$



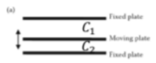
V_{out} proportional to d $\quad V_{out} = -\frac{C_f}{C}V_{in} = -\frac{C_f d}{\varepsilon_0\varepsilon_r A}V_{in}$

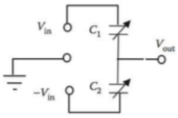


AC Bridge

- The capacitances between the plates C_1 and C_2 comprise a voltage divider circuit.
- Equivalent capacitance $\frac{C_1C_2}{C_1+C_2}$
- The output voltage V_{out} is

$$V_{out} = -V_{in} + 2V_{in} \times \frac{C_1}{C_1 + C_2} = \frac{C_1 - C_2}{C_1 + C_2}V_{in}$$





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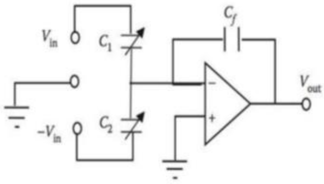
- The output voltage V_{out} is

$$V_{out} = -\frac{C_1 - C_2}{C_f}V_{in}$$

where C_1 and C_2 are the capacitances between the plates.

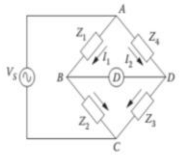


- For the balance condition:

$$V_B = V_D$$

- Therefore,

$$V_S\frac{Z_1}{Z_1 + Z_2} = V_S\frac{Z_4}{Z_3 + Z_4} \Rightarrow Z_1Z_3 = Z_2Z_4$$



- Since impedance is a complex number,

$$\begin{cases}\mathrm{Re}(Z_1Z_3) = \mathrm{Re}(Z_2Z_4)\\ \mathrm{Im}(Z_1Z_3) = \mathrm{Im}(Z_2Z_4)\end{cases}$$

- The complex impedance balance condition can also be expressed in polar form:

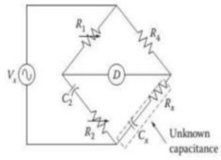
$$\begin{cases}|Z_1||Z_3| = |Z_2||Z_4|\\ \angle\theta_1 + \angle\theta_3 = \angle\theta_2 + \angle\theta_4\end{cases}$$

Comparison Bridge

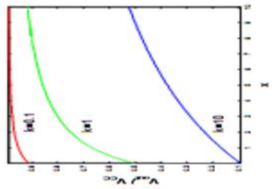
- A comparison bridge measures an unknown capacitance or inductance by comparing it with a known capacitance or inductance.
- Under the bridge balance condition:

$$Z_1Z_x = Z_2Z_4 \Rightarrow R_1\left(R_x + \frac{1}{j\omega C_x}\right) = R_4\left(R_2 + \frac{1}{j\omega C_2}\right)$$

$$\Rightarrow \begin{cases}R_x = \frac{R_2R_4}{R_1}\\ C_x = \frac{R_1C_2}{R_4}\end{cases}$$



Bridges

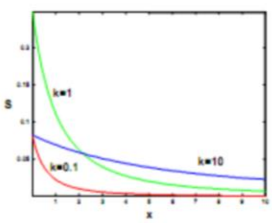


The output voltage of the circuit is

$$V_{out} = V_{CC}\frac{R_S}{R_S + R_L} = V_{CC}\frac{R_0(1+x)}{R_0(1+x) + R_0k} = V_{CC}\frac{1+x}{1+x+k}$$

- **What is the sensitivity of this circuit?**

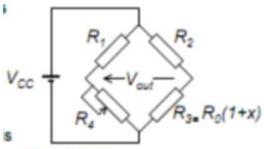
$$S = \frac{dV_{out}}{dx} = \frac{d}{dx}\left(V_{CC}\frac{1+x}{1+x+k}\right) = V_{CC}\frac{(1+x+k)-(1+x)}{(1+x+k)^2} = V_{CC}\frac{k}{(1+x+k)^2}$$



- **For which R_L do we achieve maximum sensitivity?**

$$\frac{dS}{dk} = 0 \Rightarrow \frac{d}{dk}\left(V_{CC}\frac{k}{(1+x+k)^2}\right) = 0 \Rightarrow \frac{(1+x+k)^2 - k2(1+x+k)}{(1+x+k)^2} = 0 \Rightarrow \mathbf{k = 1 + x}$$

- This is, the sensitivity is maximum when $R_L = R_S$



- Null mode
  - R_4 adjusted until the balance condition is met:

$$V_{out} = 0 \Leftrightarrow R_3 = R_4 \frac{R_2}{R_1}$$

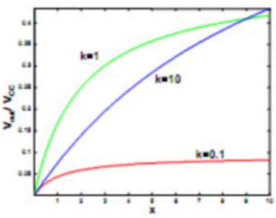
    - Advantage: measurement is independent of fluctuations in V_{CC}
- Deflection mode
  - The unbalanced voltage V_{out} is used as the output of the circuit

$$V_{out} = V_{CC}\left(\frac{R_3}{R_2+R_3} - \frac{R_4}{R_3+R_4}\right)$$

    - Advantage: speed

- **Assumptions**
  - Want to measure sensor fractional resistance changes $R_S=R_0(1+x)$
  - Bridge is operating near the balance condition:

$$k = \frac{R_1}{R_4} = \frac{R_2}{R_0}$$



- **The output voltage becomes**

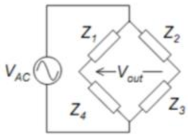
$$V_{out} = V_{CC}\left(\frac{R_0(1+x)}{R_0k+R_0(1+x)} - \frac{R_4}{R_4k+R_4}\right) =$$
$$= V_{CC}\left(\frac{(1+x)}{k+(1+x)} - \frac{1}{k+1}\right) = V_{CC}\frac{kx}{(1+k)(1+k+x)}$$

What is the sensitivity of the Wheatstone bridge?

$$S = \frac{dV_{out}}{dx} = V_{CC}\frac{d}{dx}\left(\frac{kx}{(1+k)(1+k+x)}\right) =$$
$$= V_{CC}\frac{k(1+k)(1+k+x) - kx(1+k)}{(1+k)^2(1+k+x)^2} =$$
$$= V_{CC}\frac{k}{(1+k+x)^2}$$

- **The balance condition becomes**

$$\frac{Z_1}{Z_4} = \frac{Z_2}{Z_3}$$



- which yields two equalities, for real and imaginary components

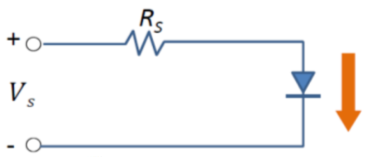
$$R_1R_3 - X_1X_3 = R_2R_4 - X_2X_4$$
$$R_1X_3 + X_1R_3 = R_2X_4 + X_2R_4$$

Interfacing Circuits_transistor amplifiers

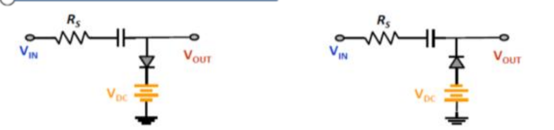
Diodes

Approximation:

- Forward Bias:
  If $V_D > V_\gamma$ (~ 0.6-0.7 V)
  ⟹ Diode conducts (short circuit)
- Reverse Bias:
  If $V_Z < V_D < V_\gamma$
  ⟹ Diode does not conduct (open circuit)

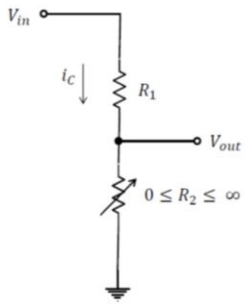


Current: $\frac{V_S - V_\gamma}{R_S}$



$V_{OUT} = V_{IN} - V_{PEAK} + V_{DC}$ $\quad$ $V_{OUT} = V_{IN} + V_{PEAK} - V_{DC}$

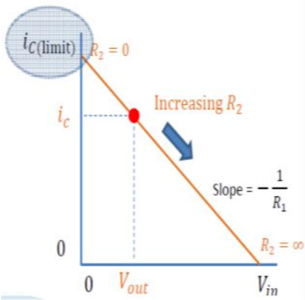
- Load voltage equals V_Z if the Zener diode is in the reverse breakdown region: $i_L = \frac{V_Z}{R_L}$
- Load current comes from KCL: $i_L = i_S - i_Z$
- Source current is: $i_S = \frac{V_S - V_Z}{R_S}$
- Zener diode is usually rated by its maximum allowable power dissipation: $P_{Z,max} = i_{Z,max} \cdot V_Z$

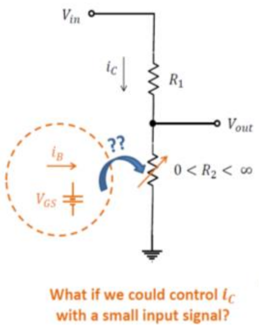


$$V_{out} = \frac{R_2}{R_1 + R_2} V_{in}$$

$$V_{out} = V_{in} - i_C R_1$$

V_{out} is dependent on i_C, which in turn is controlled by adjustments to R_2

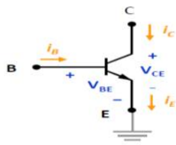




$$V_{out} = V_{in} - (i_B\beta)R_1$$

BJT

$$i_E = i_C + i_B$$
$$\frac{i_E}{i_C} = 1 + \frac{i_B}{i_C}$$
$$\frac{1}{\alpha_{dc}} = 1 + \frac{1}{\beta_{dc}}$$



$$\beta_{dc} = \frac{\alpha_{dc}}{1-\alpha_{dc}} \quad \leftrightarrow \quad \alpha_{dc} = \frac{\beta_{dc}}{\beta_{dc}+1}$$

Three operation modes:

Active Linear – Current Amplification
$i_C = i_B\,\beta$

Cutoff – Open Switch (no collector current)
$i_C \approx 0, \quad R_{CE} \approx \infty$

Saturation – Closed Switch ($V_{CE} \to 0$)
$i_C \approx i_{C(\text{limit})}, \quad R_{CE} \approx 0$

Active Linear – Current Amplification
$(V_{BE} = V_\gamma \ \& \ V_{CE} > V_\gamma) \Longrightarrow i_C = i_B\,\beta;$

➢Power dissipated: $P = i_C \cdot V_{CE}$

Cutoff – No collector current flow.

$$V_{BE} < V_Y \Rightarrow i_B = 0 \Rightarrow i_C \approx 0;\ V_{CE} \geq 0$$

Saturation – Closed Switch.

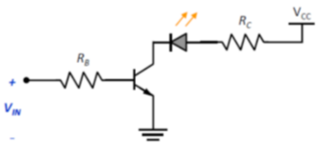
$$\left(V_{BE} = V_Y \ \&\ i_B > \frac{i_{C\,(\text{limit})}}{\beta}\right) \Rightarrow V_{CE} = V_{SAT}$$

- **Point A** [$i_B \approx 0$ or small V_{IN} (< 0.6 V)]
  - transistor is *cutoff*
  - $i_C \approx i_E \approx 0 \Rightarrow V_{OUT} \approx VCC$
  - Switch is open!
- **Point B** [$i_B > i_{B(sat)}$ or large V_{IN} (> 0.7 V)]
  - transistor is *saturated*.
  - $V_{OUT} = V_{CE\,(sat)} \approx 0.2\ V$ (very small!)
  - Switch is closed!

$$i_B = \frac{V_{IN} - V_{BE(SAT)}}{R_B};\quad i_C \approx \frac{V_{CC} - V_{CE(SAT)}}{R_L}$$

$$i_B \approx \frac{i_{C(\text{limit})}}{10}$$

- turn-ON time $t_{ON} = t_D + t_R$
- turn-OFF time $t_{OFF} = t_S + t_F$



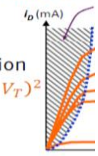
$R_c = (V_{cc} - V_{LED})/I_{LED} = (1$

$R_B = (V_{in} - 0.7)/(I_{LED}/10)$

MOSFET

Four operation region:

- Cutoff state – Transistor is turned OFF
  $V_{GS} < V_T \Rightarrow i_D \approx 0;\quad V_{DS} \approx V_{DD}$
- Ohmic state – Linear (or triode) region
  $(V_{GS} > V_T \ \&\ V_{DS} < V_{GS} - V_T \ll V_{DD}) \Rightarrow i_D \approx V_{DD}/R_D$;
  - ➢ i_D is controlled by the drain circuit
  - ➢ From D to S can be viewed as closed with a voltage-controlled (small) resistor
- Constant current – Saturation (or active) region
  $(V_{GS} > V_T \ \&\ V_{DS} > V_{GS} - V_T) \Rightarrow i_D \propto (V_{GS} - V_T)^2$
  - ➢ i_D is controlled by the gate-source voltage
  - ➢ Power dissipated: $P = i_D \cdot V_{DS}$
- Breakdown – Transistor will get VERY HOT!

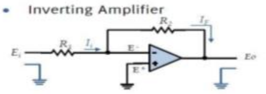


- **Point A** ($V_{IN} < V_T$)
  - ➢ transistor is cutoff
  - ➢ $i_D \approx i_S \approx 0 \Rightarrow V_{OUT} \approx V_{DD}$
  - ➢ Switch open!
- **Point B** ($V_{IN} > V_T$)
  - ➢ transistor is in Ohmic region
  - ➢ $V_{OUT} = V_{DD} - V_{DS} = V_{DD} - i_D(V_{G1}) \cdot R_D$
  - ➢ Switch closed!

OPAMPS and its applications

$$E_O = G_O(E^+ - E^-)$$

- Inverting Amplifier

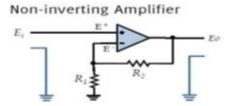


$$\frac{E_i - E^-}{R_1} = I_i = I_F = \frac{E^- - E_O}{R_2}$$

$$E^- = E^+ = 0$$

$$\frac{E_O}{E_i} = -\frac{R_2}{R_1} \Rightarrow E_O = -\frac{R_2}{R_1}E_i$$

- Non-inverting Amplifier



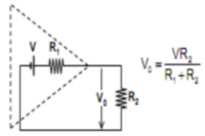
$$E_i = \left(\frac{R_1}{R_1 + R_2}\right)E_O \Rightarrow E_O = \left(\frac{R_1 + R_2}{R_1}\right)E_i$$

$$E_O = \left(1 + \frac{R_2}{R_1}\right)E_i$$

efficiency in terms of power

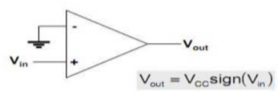
$$\eta = \frac{P_{out}}{P_{in}}$$

Input impedance is the ratio of input voltage to input current

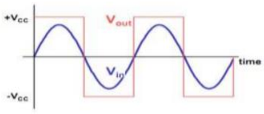


$$Z_{in} = \frac{V_{in}}{I_{in}}$$

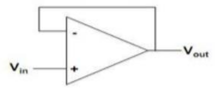
- **Voltage comparator**



$V_{out} = V_{CC}\text{sign}(V_{in})$

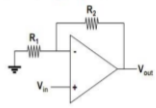


- **Voltage follower**
  - What is the main use of this circuit?
    - Buffering



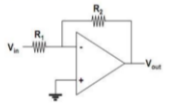
$V_{out} = V_{in}$

- **Non-inverting amplifier**



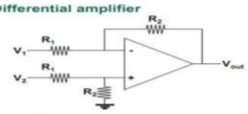
$V_{out} = \left(1 + \frac{R_2}{R_1}\right)V_{in}$

- **Inverting amplifier**



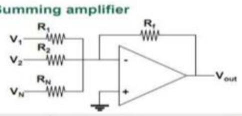
$V_{out} = -\frac{R_2}{R_1}V_{in}$

- **Differential amplifier**



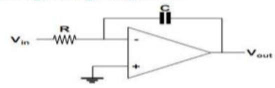
$V_{out} = \frac{R_2}{R_1}(V_2 - V_1)$

- **Summing amplifier**



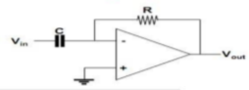
$V_{out} = -\left(V_1\frac{R_f}{R_1} + V_2\frac{R_f}{R_2} + \cdots + V_N\frac{R_f}{R_N}\right)$

- **Integrating amplifier**



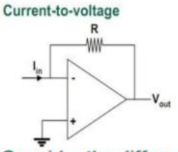
$V_{out} = -\frac{1}{j\omega CR}V_{in} = -\frac{1}{RC}\int V_{in}dt$

- **Differentiating amplifier**



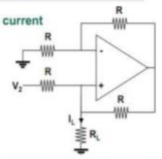
$V_{out} = -\frac{R}{\frac{1}{j\omega C}}V_{in} = -RC\frac{dV_{in}}{dt}$

- **Current-to-voltage**



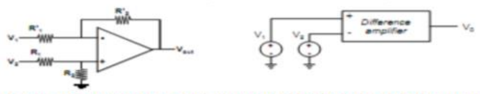
$V_{out} = -I_{in}R$

- **Voltage to current**



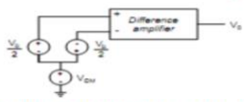
$I_L = \frac{V_2}{R}$

- **Consider the difference amplifier we saw in the previous lecture**



- **We define COMMON-MODE and DIFFERENCE-MODE voltage as**

$$V_{CM} = \frac{V_2 + V_1}{2}$$
$$V_D = V_2 - V_1$$



- **As a result of a mismatch in the resistors ($R'_k \neq R_k$), the differential inputs may not have the same gain**

$$V_o = G(V_2 - V_1) \overset{R'_k \neq R_k}{=} G_2V_2 - G_1V_1 = G_2\left(-\frac{V_D}{2} + V_{CM}\right) - G_1\left(\frac{V_D}{2} + V_{CM}\right) =$$
$$= -V_D\left(\frac{G_2 + G_1}{2}\right) + V_{CM}(G_2 - G_1) = -V_DG_D + V_{CM}G_{CM}$$

- **We define COMMON-MODE REJECTION RATIO as**

$$CMRR = 20\log_{10}\left(\frac{G_D}{G_{CM}}\right) = 20\log_{10}\left(\frac{G_2 + G_1}{2(G_2 - G_1)}\right)$$

$V_{out} = mV_{in} + V_0$ $\quad P = I^2R \quad$ $V_{out} = mR_s + V_0$

$\therefore R_t = R_o\alpha_o t + R_o = R_o(1 + \alpha_o t)$

$P = P_D\Delta T \quad I = [P/R]^{1/2} ;\ V = IR \quad V_{max} = \sqrt{PR} \quad V_{TH} = IR_1 \quad R_2 = V/I$

$$i = \frac{Vin - Va}{R1} = \frac{Va - Vout}{R2}$$
$$Va = 0$$
$$\frac{Vin}{R1} = -\frac{Vout}{R2}$$

$$Vin = -\frac{R1}{R2}Vout$$